

RETRACTS AND RETRACTILE SUBCOMPLEXES

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§1. INTRODUCTION

IN THIS PAPER we are concerned with finding general criteria for determining when a subcomplex A of a CW -complex K is, or is not, a retract of K . In particular, we seek to generalize the classical theorem on the n -cell and its boundary:

(1.1) S^{n-1} is not a retract of I^n .

A frequently encountered proof of (1.1) arises from the fact that $S(S^{n-1}) = S^n$ and $S(I^n) = I^{n+1}$, where S denotes the suspension, and employs an inductive argument based on these standard results of (reduced) cohomology theory:

(1.2) There is a natural isomorphism $\tilde{H}^k(K) \cong \tilde{H}^{k+1}(SK)$ for each integer $k \geq 0$.

(1.3) If A is a retract of K , then the inclusion map $i: A \rightarrow K$ induces an epimorphism $i^*: \tilde{H}^k(K) \rightarrow \tilde{H}^k(A)$ for all $k \geq 0$.

This approach to the proof of (1.1) suggests the question: if the subcomplex A is not a retract of the CW -complex K , when is it also true that SA is not a retract of SK ? Using a proof based on (1.2) and (1.3), one can easily show that this is the case when

(a) K is contractible.

In §4 it will be proved that the CW -pairs (K, A) with the property, A is a retract of K if and only if SA is a retract of SK , also include the cases:

(b) each component of A is an H -space;

(c) A is m -connected and $\dim(K \setminus A) \leq 2m + 1$.

All three of these cases are corollaries of our general result, Theorem (4.2).

A second generalization of (1.1) can be based on the observation that for nonnegative integers j and k , $I^j * S^k = I^{j+k+1}$ and $S^{j-1} * S^k = S^{j+k}$, where $K * L$ denotes the join of K and L . Thus we ask, if A is not a retract of K , for what CW -complexes L is it true that $A * L$ is not a retract of $K * L$? An answer to this question, in terms of the cohomology of the pair (K, A) , is given by Theorem (5.1). We also show in Theorem (3.1) that if SA is a retract of SK , then $A * L$ is a retract of $K * L$ for every CW -complex L . Thus the second question posed is closely associated with the first.

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Yet another generalization of (1.1) can be based on the fact that for positive integers j and k , if $n = j + k$, then $(I^j \times S^{k-1}) \cup (S^{j-1} \times I^k)$ is the boundary, S^{n-1} , of $I^j \times I^k = I^n$. Then we ask, if A is not a retract of K , for what CW -pairs (L, B) is it true that $(K \times B) \cup (A \times L)$ is then not a retract of $K \times L$? In Theorem (6.2) we show that this will be the case whenever $A * (L/B)$ is not a retract of $K * (L/B)$. Thus we are led back again to the first two questions.

The organization of the material is as follows. Basic definitions and conventions are to be found in §2. In §3 we establish various general properties of retractile subcomplexes, which are those subcomplexes A of a CW -complex K such that SA is a retract of SK , cf. [9], [4]. The questions we have posed above, as well as several others suggested by them, are dealt with in turn in §§4, 5 and 6. We also give examples to show that the theorems that answer these questions are "best possible".

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§2. PRELIMINARIES

All spaces will be Hausdorff spaces with base points that we will usually denote $*$; all maps and homotopies will be assumed to preserve base points. If $W \subset X$ is a subspace containing the base point of X , then X/W will denote the space obtained by collapsing W to a point, which we take as the base point of X/W . We use $[X, W; Y]$ to denote the set of homotopy classes rel W of maps $f: X, W \rightarrow Y, *$; the element of $[X, W; Y]$ determined by the map f is denoted $[f]$. The constant map of X into Y will be denoted 0 and the identity map of X into itself will be denoted 1_X , or simply 1 .

Given maps $g: H, A \rightarrow K, B$ and $r: X \rightarrow Y$, we have the induced functions $g^*: [H, A; X] \rightarrow [K, B; X]$, $r_*: [H, A; X] \rightarrow [H, A; Y]$ defined by $g^*[f] = [fg]$, $r_*[f] = [rf]$ for $[f] \in [H, A; X]$; the induced functions satisfy the equality

$$(2.1) \quad g^*r_* = r_*g^*: [H, A; X] \rightarrow [K, B; Y].$$

Given maps $h: L, C \rightarrow K, B$ and $s: Y \rightarrow Z$ then, with g and r as above,

$$(2.2) \quad (sr)_* = s_*r_* \text{ and } (gh)^* = h^*g^*.$$

The only functions $\beta: [K, B; X] \rightarrow [H, A; Y]$ between homotopy sets that we consider in this paper are those which arise in a natural manner, and so satisfy $\beta[0] = [0]$. Such a function β will be said to be *0-monic* if the subset $\beta^{-1}[0] \subset [K, B; X]$ consists of the single element $[0]$. When β is a one-to-one mapping into $[H, A; Y]$, we say that β is *monic*; when β sends $[K, B; X]$ onto $[H, A; Y]$, we say β is *epic*; β will be called an *equivalence* whenever it is both epic and monic. The terms *monomorphism*, *epimorphism* and *isomorphism* will be reserved for the corresponding cases in which β preserves a natural algebraic structure on the homotopy sets. If $\gamma: [L, C; W] \rightarrow [K, B; X]$ is another function between homotopy sets, we say that the sequence

$$[L, C; W] \xrightarrow{\gamma} [K, B; X] \xrightarrow{\beta} [H, A; Y]$$

is *exact* at $[K, B; X]$ if $\gamma([L, C; W]) = \beta^{-1}[0]$. Observe that if $\beta\gamma$ is 0-monic, then γ is 0-monic; as usual, $\beta\gamma$ epic implies β is epic and $\beta\gamma$ monic implies γ is monic.

Henceforward, we reserve the upper case letters in the first half of the alphabet to represent *CW*-complexes; we shall always assume that the base point is a subcomplex. Pairs (K, A) of such letters will denote *CW*-pairs, where A is a subcomplex whose base point coincides with the base point of K . Whenever a topological product $K \times L$ of *CW*-complexes is considered, we shall assume that it is again a *CW*-complex. This is the case, for instance, when K and L are countable complexes [12]. The base point of $K \times L$ is the subcomplex $(*, *)$. If (K, A) is a *CW*-pair and $p: K, A \rightarrow K/A$ is the identification map onto the quotient complex, then in view of the homotopy extension theorem for *CW*-complexes, the induced function $p^*: [K/A; X] \rightarrow [K, A; X]$ is an equivalence for every space X . Indeed, if B is a subcomplex of A , then the identification map $q: K, A \rightarrow K/B, A/B$ also induces a natural equivalence $q^*: [K/B, A/B; X] \rightarrow [K, A; X]$.

Let K and L be disjoint *CW*-complexes. The *wedge* of K and L is the space $K \vee L$ obtained by identifying the base points of K and L ; alternatively, $K \vee L$ is the subcomplex of $K \times L$ consisting of all points (x, y) such that either $x = *$ or $y = *$. The *smash product* of K and L is the identification space $K \wedge L = (K \times L)/(K \vee L)$. Let I denote the closed unit interval $[0, 1]$. We define the *join* of K and L to be the space $K * L$ obtained from $K \times L \times I$ by identifying the points $(x, y, 0)$ and $(x, z, 0)$ for all $y, z \in L$ and identifying points $(w, y, 1)$ and $(x, y, 1)$ for all $w, x \in K$. The base point of $K * L$ may be taken to be $(*, *, 1/2)$.

Let X be an arbitrary space. The space of maps $f: L \rightarrow X$ with the compact open topology will be denoted X^L ; its base point is the constant map $*^L$. With K and L *CW*-complexes, we define the *adjoint equivalence* $\alpha_L: [K \wedge L; X] \rightarrow [K; X^L]$ to be the function induced by the natural homeomorphism of $X^{K \wedge L}$ with $(X^L)^K$; that is, α_L is induced by the correspondence between maps $f: K \wedge L \rightarrow X$ and maps $f': K \rightarrow X^L$, where $(f'(x))(y) = f(x, y)$ for $x \in K$ and $y \in L$.

With $S^0 \subset I$ the boundary 0-sphere, we take S^1 to be the identification space I/S^0 and we define the (*reduced*) *suspension functor* S by $SK = K \wedge S^1$. Given a map $f: K \rightarrow L$ we define $Sf: SK \rightarrow SL$ by $(Sf)(x, t) = (f(x), t)$ for $x \in K$ and $t \in I$. The *loop functor* Ω is defined correspondingly: $\Omega X = X^{S^1}$; given a map $f: X \rightarrow Y$, we define $\Omega f: \Omega X \rightarrow \Omega Y$ by $((\Omega f)(\omega))(t) = f(\omega(t))$ for $t \in I$ and $\omega \in \Omega X$. The homotopy sets $[SK; X]$, $[K; \Omega X]$ possess natural group structures [2] and the adjoint equivalence $\alpha: [SK; X] \rightarrow [K; \Omega X]$ is a natural isomorphism with respect to these group structures.

We note here some useful relations involving smash, suspension and join. Here, and throughout the rest of the paper, we use the symbol " \simeq " to denote homotopy equivalence.

LEMMA (2.3). *Let (K, A) and (L, B) be *CW*-pairs. Then*

- (a) $K * L \simeq S(K \wedge L) \simeq K \wedge (SL) \simeq (SK) \wedge L$.
- (b) $S(K \times L) \simeq S(K \wedge L) \vee SK \vee SL \simeq (K * L) \vee SK \vee SL$.
- (c) $(K \times L)/(K \times B \cup A \times L)$ is homeomorphic to $(K/A) \wedge (L/B)$.
- (d) $(K \wedge L)/(A \wedge L)$ is homeomorphic to $(K/A) \wedge L$.

For (a), see [1, p. 11] and observe that $(K \wedge L) \wedge M \simeq K \wedge (L \wedge M)$ and $L \wedge M \simeq M \wedge L$

hold for any CW -complex M . For (b), see [17, p. 310]. Statements (c) and (d) follow easily from the definitions.

Let (K, A, B) be a CW -triple and let $i : A, B \rightarrow K, B$ and $j : K, B \rightarrow K, A$ denote the inclusions. Then for each space X there is a natural exact sequence [2], [17]:

$$(2.4) \quad \dots \rightarrow [K, B; \Omega X] \xrightarrow{i^*} [A, B; \Omega X] \xrightarrow{\delta} [K, A; X] \xrightarrow{j^*} [K, B; X] \xrightarrow{i^*} [A, B; X].$$

The portion of the sequence (2.4) to the left of δ is an exact sequence of groups and homomorphisms. Recall that a space X is an H -space if there is a map $m : X \times X \rightarrow X$ such that $m(x, *) = m(*, x) = x$ for every $x \in X$. If X is a connected H -space, the entire sequence is an exact sequence of (algebraic) loops and homomorphisms and the homotopy sets to the left of δ are abelian groups [9].

Let P be a subspace of Q and let $i : P \rightarrow Q$ be the inclusion map. P is said to be a *retract* of Q if there exists a map $r : Q \rightarrow P$ such that $ri = 1 : P \rightarrow P$; such a map r is called a *retraction*. Recall that P is a retract of Q if and only if every map $f : P \rightarrow X$ can be extended over Q . From this and the homotopy extension theorem for CW -complexes one obtains

LEMMA (2.5). *Let (K, A) be a CW -pair and $i : A \rightarrow K$ be the inclusion map. Then the following statements are equivalent:*

- (a) A is a retract of K .
- (b) $i^* : [K; A] \rightarrow [A; A]$ is epic.
- (c) $i^* : [K; X] \rightarrow [A; X]$ is epic for every space X .

For (2.5) we have the following partial dual statement: If A is a retract of K then the inclusion map $i : A \rightarrow K$ induces a monic $i_* : [X; A] \rightarrow [X; K]$ for every CW -complex X . In particular, the dual of (1.3) holds:

$$(2.6) \quad \text{If } A \text{ is a retract of } K, \text{ the inclusion map } i : A \rightarrow K \text{ induces a monomorphism } i_* : \pi_k(A) \rightarrow \pi_k(K) \text{ for each } k \geq 0.$$

§3. RETRACTILE SUBCOMPLEXES

I. M. James [9] defined the concept of *retractile* subcomplex as follows: a subcomplex A is retractile in a complex K provided that K/A is contractible in CK/A , where CK denotes the (reduced) cone over K . It is clear from this definition that if (L, B) is a CW -pair and $(K, A) \simeq (L, B)$, then A is retractile in K whenever B is retractile in L . Thus "retractile", as well as "retract", is a homotopy invariant of CW -pairs. In this section, after we give a number of characterizations of retractile subcomplexes, we shall establish several of their general properties. Our first result provides characterizations of retractile subcomplexes in terms of retracts.

THEOREM (3.1). *The following restrictions on a CW -pair (K, A) are equivalent:*

- (a) A is retractile in K .
- (b) SA is a retract of SK .
- (c) $A * L$ is a retract of $K * L$ for every CW -complex L .

Proof. The assertion (a) \Leftrightarrow (b) is due to Eckmann and Hilton [4, Theorem 2.2]. The implication (c) \Rightarrow (b) follows immediately on taking $L = S^0$; thus it remains only to show (b) \Rightarrow (c). But if $r : SK \rightarrow SA$, is a retraction then $r \wedge 1_L : (SK) \wedge L \rightarrow (SA) \wedge L$ is also a retraction, and the result now follows from the homotopy equivalence $(K * L, A * L) \simeq ((SK) \wedge L, (SA) \wedge L)$ of (2.3a).

COROLLARY (3.2). *A is retractile in K if and only if $A \wedge L$ is retractile in $K \wedge L$ for every CW-complex L.*

Next, we characterize subcomplexes in terms of mappings of homotopy sets.

THEOREM (3.3). *The following statements are equivalent:*

- (a) *A is retractile in K.*
- (b) *The inclusion $i : A \rightarrow K$ induces an epimorphism $i^* : [K; \Omega X] \rightarrow [A; \Omega X]$ for every space X.*
- (c) *The inclusion $j : K \rightarrow K, A$ induces a 0-monic $j^* : [K, A; X] \rightarrow [K; X]$ for every space X.*
- (d) *The identification map $q : K \rightarrow K/A$ induces a 0-monic $q^* : [K/A; X] \rightarrow [K; X]$ for every space X.*

Proof. By (3.1b) and (2.5c), A is retractile in K if and only if $(Si)^* : [SK; X] \rightarrow [SA; X]$ is epic for every space X. Now (a) \Leftrightarrow (b) follows from this by means of the adjoint equivalence. The implication (a) \Rightarrow (c) is due to James [9, Lemma 3.2]; indeed, (b) \Leftrightarrow (c) is immediate from the exact sequence (2.4). Finally, (c) \Leftrightarrow (d) results from the natural equivalence between $[K, A; X]$ and $[K/A; X]$.

The following result generalizes (3.3b).

THEOREM (3.4). *Let (K, A) be a CW-pair and let $i : A \rightarrow K$ be the inclusion map. Then A is retractile in K if and only if $i^* : [K; Y] \rightarrow [A; Y]$ is an epimorphism for every connected H-space Y.*

Proof. First we show that we may restrict the connected H-spaces to be CW-complexes without loss of generality. For if L is the geometric realization of the total singular complex of Y, then L is a CW-complex which is an H-space, and the canonical map $j : L \rightarrow Y$ induces a one-to-one correspondence $j_* : \pi_n(L) \rightarrow \pi_n(Y)$, $n = 0, 1, \dots$, [14, Theorem 4]. Hence by [9, Lemma 3.4], the function $j_* : [R; L] \rightarrow [R; Y]$ is an equivalence for every CW-complex R. Now let Y be a connected CW-complex that is an H-space. Then Y is dominated by ΩSY , [8], so there are maps $h : Y \rightarrow \Omega SY$ and $k : \Omega SY \rightarrow Y$ such that kh is homotopic to 1_Y . We note that $k_* : [R; \Omega SY] \rightarrow [R; Y]$ is epic for every space R. By (2.1) the diagram

$$\begin{array}{ccc} [K; \Omega SY] & \xrightarrow{i^*} & [A; \Omega SY] \\ k_* \downarrow & & \downarrow k_* \\ [K; Y] & \xrightarrow{i^*} & [A; Y] \end{array}$$

is commutative. Now suppose A is retractile in K. Then by (3.3b) the function $i^* : [K; \Omega SY] \rightarrow [A; \Omega SY]$ is epic. Since the functions k_* are epic it follows that

$i^* : [K; Y] \rightarrow [A; Y]$ is also epic. Conversely, suppose $i^* : [K; Y] \rightarrow [A; Y]$ is epic for every connected H -space Y . Since each path component of ΩX is an H -space for every space X , it then follows that $i^* : [K; \Omega X] \rightarrow [A; \Omega X]$ is epic, so A is retractile in K by (3.3b).

COROLLARY (3.5). *A is retractile in K if and only if every map from A into a connected H -space can be extended over K .*

That the H -space Y must be connected for (3.4) and (3.5) to hold is illustrated by the following example. Let $(K, A) = (S^2 \times S^2 \cup P, S^2 \vee S^2 \cup P)$, where $\cup P$ denotes the disjoint union with a one-point space P which we take as the basepoint. Let $Y = S^2 \cup P$. Then the map $m : Y \times Y \rightarrow Y$ defined by $m(S^2 \times S^2) = P$ and $m(x, P) = m(P, x) = x$ for all $x \in S^2$ yields an H -structure on Y . If $f : A \rightarrow Y$ is the map whose restriction to $S^2 \vee S^2 \subset A$ is the folding map into $S^2 \subset Y$, then f cannot be extended to a map of K into Y ; for S^2 is not an H -space. However, by (3.6) below, A is retractile in K ; but as the example shows, A is not a retract of K . Indeed, (3.6) establishes a large class of retractile subcomplexes that are usually not retracts.

THEOREM (3.6) [9, Lemma 3.1]. *If A_1, \dots, A_n are retracts of K and if there are retractions $r_i : K \rightarrow A_i$ such that $r_i(A_j) \subset A_j$, $i, j = 1, \dots, n$, then $A = A_1 \cup \dots \cup A_n$ is retractile in K .*

The remainder of this section is devoted to establishing a number of properties of retractile subcomplexes that are also properties of retracts; that is, each of propositions (3.7)–(3.9) below is equally valid with “retractile” replaced by “retract”.

PROPOSITION (3.7). *Let (K, A, B) be a CW-triple.*

- (a) *If B is retractile in A and A is retractile in K , then B is retractile in K .*
- (b) *If B is retractile in K , then B is retractile in A .*
- (c) *If A is retractile in K , then A/B is retractile in K/B .*

The proofs are straightforward from the corresponding statements for retracts by applying (3.1b).

If A and B are retracts of a complex K , it is false in general that $A \cup B$ or $A \cap B$ is a retract of K . Indeed, we may have A, B and $A \cup B$ all retracts of K , while $A \cap B$ fails to be a retract; for example, let A, B be the upper and lower hemispheres of $S^n = K$. Our next result deals with similar considerations for retractile subcomplexes.

PROPOSITION (3.8). *Let A and B be subcomplexes of a CW-complex K .*

- (a) *If $A \cup B$ is retractile in K , but B is not retractile in K , then $A \cap B$ is not retractile in A .*
- (b) *If $K = A \cup B$ and if $A \cap B$ is retractile in A and in B , then $A \cap B$ is retractile in K .*

Proof. (a) By (3.5) there is a connected H -space Y and a map $f : B \rightarrow Y$ that cannot be extended over K , so $f|_{A \cap B}$ cannot be extended over A . For otherwise there is a map $g : A \cup B \rightarrow Y$, with $g|_B = f$, such that g cannot be extended over K , a contradiction.

(b) Any map of $A \cap B$ into a connected H -space can be extended over A and over B , hence it can be extended over $A \cup B$.

One can easily show that a subspace W of a space X is a retract of X if and only if

$W \times Y$ is a retract of $X \times Y$, where Y is an arbitrary nonempty space. Next we prove the corresponding assertion for retractile subcomplexes.

PROPOSITION (3.9). *Let (K, A) be a CW -pair and let L be a CW -complex. Then A is retractile in K if and only if $A \times L$ is retractile in $K \times L$.*

Proof. By (2.3b) we have $(S(K \times L), S(A \times L)) \simeq (K * L \vee SK \vee SL, A * L \vee SA \vee SL)$.

If A is retractile in K , then by (3.1) there are retractions $q : K * L \rightarrow A * L$ and $r : SK \rightarrow SA$; hence we have a retraction $q \vee r \vee 1 : K * L \vee SK \vee SL \rightarrow A * L \vee SA \vee SL$. Therefore $S(A \times L)$ is a retract of $S(K \times L)$, so $A \times L$ is retractile in $K \times L$. Conversely, if $A \times L$ is retractile in $K \times L$, there is a retraction $p : K * L \vee SK \vee SL \rightarrow A * L \vee SA \vee SL$, and p induces a retraction of SK onto SA . Therefore A is retractile in K .

§4. WHEN IS A RETRACTILE SUBCOMPLEX A RETRACT?

In this section we seek general criteria for determining when a given retractile subcomplex A of a CW -complex K is, or is not a retract of K . We consider the negative question first.

If K and L are CW -complexes, then it follows from (3.6) that $K \vee L$ is retractile in $K \times L$. Under special conditions it may indeed happen that $K \vee L$ is also a retract of $K \times L$; an example of this is constructed in §5 of [5]. However, $K \vee L$ is usually not a retract of $K \times L$ and, as we show next, $K \vee K$ is a retract of $K \times K$ only in trivial cases.

THEOREM (4.1). *If K is a connected, noncontractible CW -complex, then $K \vee K$ is not a retract of $K \times K$.*

Proof. Suppose K is a connected CW -complex and suppose that $K \vee K$ is a retract of $K \times K$. One can easily show that $(K \vee K) \vee (K \vee K)$ is then a retract of $(K \vee K) \times (K \vee K)$. Therefore by (2.5) the inclusion map induces an epic function

$$[(K \vee K) \times (K \vee K); K \vee K] \rightarrow [(K \vee K) \vee (K \vee K); K \vee K],$$

so by [15, Theorem 2.1], $K \vee K$ is an H -space. Then it follows immediately from [5, Theorem 1.5] that K is contractible.

The question of when a retractile subcomplex is also a retract is dealt with in our next theorem, the main result of this section.

THEOREM (4.2). *Let A be a connected, retractile subcomplex of K and suppose there is an H -space X and a map $f : A \rightarrow X$ such that $f_* : \pi_k(A) \cong \pi_k(X)$ for all $k < n$, where n is a positive integer or ∞ . If $\tilde{H}^{k+1}(K, A; \pi_k(A)) = 0$ for all $k \geq n$, or if $\tilde{H}^{k+1}(K; \pi_k(A)) = 0$ for all $k \geq n$, then A is a retract of K .*

Proof. Without loss of generality we may assume, as in the proof of (3.4), that the H -space X is a CW -complex. Let B be the space obtained from A by adjoining cells in dimensions greater than n so that $\pi_k(B) = 0$ for all $k \geq n$. Then B has the homotopy type of a space in a Postnikov decomposition for X ; hence B is an H -space [10, Corollary 3.1]. Then by (3.5) the inclusion map $h : A \rightarrow B$ can be extended over K to a map $g' : K \rightarrow B$; further, by the cellular approximation theorem [6, p. 98, Theorem 1.8], there is a cellular

map $g : K \rightarrow B$ such that $g \simeq g'(\text{rel } A)$. Thus $g(K^{(n)} \cup A) \subset (B^{(n)} \cup A) = A$, where $K^{(n)}$, $B^{(n)}$ are the n -skeleta of K , B , respectively. Let $r' : K^{(n)} \cup A \rightarrow A$ be the partial retraction induced by g . If $\tilde{H}^{k+1}(K, A; \pi_k(A)) = 0$ for all $k \geq n$, then the obstructions to extending r' over K to a retraction $r : K \rightarrow A$ vanish; cf. [7, p. 182, Proposition 6.6]. Therefore A is a retract of K . If $\tilde{H}^{k+1}(K; \pi_k(A)) = 0$ for all $k \geq n$, then also $\tilde{H}^{k+1}(K, A; \pi_k(A)) = 0$ whenever $k \geq n$; for when A is retractile in K , the inclusion map $j : K \rightarrow (K, A)$ induces monomorphisms $j_* : \tilde{H}^k(K, A) \rightarrow \tilde{H}^k(K)$ for all k [9, p. 164]. Therefore A is a retract of K in this case also.

We remark that (4.2) can be further generalized to retractile subcomplexes A that are not necessarily connected, provided there are H -spaces X_i and maps $f_i : A_i \rightarrow X_i$ such that $f_{i*} : \pi_k(A_i) \cong \pi_k(X_i)$, $k < n$, for each path component A_i of A . This generalization is implicit in the corollaries listed below.

COROLLARY (4.3). *If A is retractile in K and if each component of A is an H -space, then A is a retract of K .*

(This also follows from (3.4).)

COROLLARY (4.4). *If A is an m -connected, retractile subcomplex of K and if $\dim(K \setminus A) \leq 2m + 1$, then A is a retract of K .*

Proof. The case $m = 0$ is trivial. Let $m > 0$ and let B be the CW -complex obtained from A by adjoining cells in dimensions greater than $2m + 1$ so that $\pi_k(B) = 0$ for all $k \geq 2m + 1$. Then B is a connected H -space [11, Theorem (5.1)], and the inclusion map $j : A \rightarrow B$ induces isomorphisms of the homotopy groups in dimensions less than $2m + 1$. Therefore A is a retract of K .

To see that (4.4), and hence also (4.2), is "best possible", we need only consider the example: $(K, A) = (S^n \times S^n, S^n \vee S^n)$. By (3.6), A is retractile in K , and by (4.1), A is not a retract of K .

Consider the CW -triple (K, A, B) , where $K = S^1 \times S^1 \times S^1$, $A = K^{(2)} = (S^1 \times S^1 \times *) \cup (S^1 \times * \times S^1) \cup (* \times S^1 \times S^1)$, and $B = K^{(1)} = S^1 \vee S^1 \vee S^1$. As an application of (4.4) and of some of the results of §3 we shall show that $K/B \simeq A/B \vee S^3$, although A is not a retract of K . Let $p : A \rightarrow A/B$ and $q : K \rightarrow K/B$ be the identification maps, and let $i : A \rightarrow K$ and $j : A/B \rightarrow K/B$ be the inclusions. Then we have the commutative diagram

$$\begin{array}{ccc} \pi_2(A) & \xrightarrow{i_*} & \pi_2(K) \\ p_* \downarrow & & \downarrow q_* \\ \pi_2(A/B) & \xrightarrow{j_*} & \pi_2(K/B). \end{array}$$

We note first that K is obtained from A by adjoining a single 3-cell, I^3 , to A by means of a map on the boundary of I^3 , say $h : S^2 \rightarrow A$. Likewise, K/B results from A/B by attaching I^3 by the map $ph : S^2 \rightarrow A/B$. Since K and $A \vee S^3$ do not have the same homotopy type (by a cohomology ring argument, or by applying [5, Theorem 1.1]), it follows that h defines a nonzero element $[h] \in \pi_2(A)$. However, $i_*[h] = 0 \in \pi_2(K)$; therefore by (2.6), A is not a

retract of K . But by (3.6), A is retractile in K , so by (3.7c), A/B is retractile in K/B . Now A/B is 1-connected and $\dim(K/B) = 3$; hence by (4.4), A/B is a retract of K/B . Thus by (2.6), the homomorphism j_* of the diagram is a monomorphism, so $j_*[ph] = q_*i_*[h] = 0$ implies that $[ph] = 0$. Therefore $K/B \simeq A/B \vee S^3$; indeed it is easily seen that $K/B \simeq S^2 \vee S^2 \vee S^2 \vee S^3$.

§5. NONRETRACTS THAT ARE JOINS

In this section we consider the problem of determining conditions on a CW -pair (K, A) and on a complex B so that $A * B$ is not a retract of $K * B$. It is clear from (3.1) that A can be a nonretract of K while $A * B$ is a retract of $K * B$ for every CW -complex B . Indeed, it can happen that the homomorphism $i^* : \tilde{H}^*(K) \rightarrow \tilde{H}^*(A)$ induced by inclusion fails to be an epimorphism, yet there is a noncontractible complex B such that $A * B$ is a retract of $K * B$. At the end of this section we shall construct such an example where, in fact, K , A and B are finite complexes.

THEOREM (5.1). *If the homomorphism $i^* : \tilde{H}^n(K; \tilde{H}^k(B)) \rightarrow \tilde{H}^n(A; \tilde{H}^k(B))$ induced by the inclusion $i : A \rightarrow K$ fails to be an epimorphism for some pair (n, k) of integers, then $A * B$ is not retractile in $K * B$.*

Proof. Suppose $i^* : \tilde{H}^n(K; \tilde{H}^k(B)) \rightarrow \tilde{H}^n(A; \tilde{H}^k(B))$ is not an epimorphism. We show first that then $(i \wedge 1)^* : \tilde{H}^{n+k}(K \wedge B) \rightarrow \tilde{H}^{n+k}(A \wedge B)$ also fails to be an epimorphism. For the Künneth relations of reduced cohomology theory yield the natural direct sum decomposition

$$\tilde{H}^m(K \wedge B) \cong \sum_{n+k=m} \tilde{H}^n(K; \tilde{H}^k(B)),$$

and with respect to this decomposition, the inclusion map $(i \wedge 1) : A \wedge B \rightarrow K \wedge B$ induces the homomorphism $i^* : \tilde{H}^n(K; \tilde{H}^k(B)) \rightarrow \tilde{H}^n(A; \tilde{H}^k(B))$. (This form of the Künneth relation, together with a proof of its naturality, can easily be derived from homotopical cohomology [3] as in [16].) Now it follows from (1.3) that $A \wedge B$ is not a retract of $K \wedge B$, and by (1.2) and (1.3), $S^m(A \wedge B)$ is not a retract of $S^m(K \wedge B)$, $m = 1, 2, \dots$. But by (2.3a), $(S(K \wedge B), S(A \wedge B)) \simeq (K * B, A * B)$, so by (3.1b), $A * B$ is not retractile in $K * B$.

COROLLARY (5.2). *If the homomorphism $i^* : \tilde{H}^n(K; \tilde{H}^k(B)) \rightarrow \tilde{H}^n(A; \tilde{H}^k(B))$ induced by the inclusion $i : A \rightarrow K$ fails to be an epimorphism for some pair (n, k) of integers, then $A \wedge B$ is not retractile in $K \wedge B$.*

We conclude this section with an example of a finite CW -pair (K, A) and a finite, noncontractible complex B such that the homomorphism $i^* : \tilde{H}^*(K) \rightarrow \tilde{H}^*(A)$ induced by inclusion is not an epimorphism, yet $A * B$ is a retract of $K * B$. Let m and r be integers greater than 1, and take A to be the pseudo-projective space of homology type (Z_r, m) . Thus A is obtained from the m -sphere S^m by adjoining an $(m+1)$ -cell I^{m+1} by means of a map $h : S^m \rightarrow S^m$ of degree r on the boundary S^m of I^{m+1} . Let K be the contractible complex obtained by adjoining cells to A ; this can be done with just two cells: one of dimension $m+1$ and one of dimension $m+2$. Then $\tilde{H}^{m+1}(K) = 0$, while $\tilde{H}^{m+1}(A) \cong Z_r$; also $\tilde{H}^k(A) = 0$ for $k \neq m+1$. Now let B be the pseudo-projective space of homology type (Z_s, n) , where n and s are integers greater than 1 and such that r and s are relatively prime. In

[5, §5] it is proved that for A and B as given, the inclusion map $j: A \vee B \rightarrow A \times B$ is a homotopy equivalence. Then from the exact sequence

$$0 \rightarrow \tilde{H}^{k+1}(A * B) \rightarrow \tilde{H}^k(A \times B) \xrightarrow{j^*} \tilde{H}^k(A \vee B) \rightarrow 0$$

of [13, Lemma 2.1] together with the fact that $A * B$ is simply connected [13, Lemma 2.2], we see that the groups $\tilde{H}^k(A * B)$ vanish for all k . Therefore $A * B$ is contractible, and so $A * B$ is a retract of $K * B$.

§6. SUBCOMPLEXES THAT ARE UNIONS OF PRODUCTS

This final section is devoted to studying the question of when the subcomplex $(K \times B) \cup (A \times L) \subset K \times L$ arising from two CW -pairs, (K, A) and (L, B) , is or is not retractile. For the positive case we have

THEOREM (6.1). *Let A be retractile in K and let B be retractile in L . Then $(K \times B) \cup (A \times L)$ is retractile in $K \times L$.*

Proof. By (2.3c) and (3.3d) it suffices to show that the identification map $p: K \times L \rightarrow (K/A) \wedge (L/B)$ induces a 0-monic function $p^*: [(K/A) \wedge (L/B); X] \rightarrow [K \times L; X]$ for every space X . Writing p as a sequence of identification maps,

$$K \times L \xrightarrow{s} K \wedge L \xrightarrow{r} K \wedge (L/B) \xrightarrow{q} (K/A) \wedge (L/B),$$

we obtain the commutative diagram

$$\begin{array}{ccc} [(K/A) \wedge (L/B); X] & \xrightarrow{p^*} & [K \times L; X] \\ q^* \downarrow & & \uparrow s^* \\ [K \wedge (L/B); X] & \xrightarrow{r^*} & [K \wedge L; X]. \end{array}$$

Since A is retractile in K , then according to (3.2), $A \wedge (L/B)$ is retractile in $K \wedge (L/B)$; thus by (2.3d) and (3.3d), the identification map $q: K \wedge (L/B) \rightarrow (K/A) \wedge (L/B)$ induces a 0-monic $q^*: [(K/A) \wedge (L/B); X] \rightarrow [K \wedge (L/B); X]$ for every space X . Similarly, $r^*: [K \wedge (L/B); X] \rightarrow [K \wedge L; X]$ is always 0-monic. Finally, the identification $s: K \times L \rightarrow (K \times L)/(K \vee L) = K \wedge L$ induces a 0-monic; for $K \vee L$ is retractile in $K \times L$ by (3.6). Therefore p^* is a composition of 0-monic functions, so p^* is 0-monic.

Next we ask, if A is not retractile in K , when can we conclude that $(K \times B) \cup (A \times L)$ is not retractile in $K \times L$? At the end of this section we construct an example of finite CW -pairs (K, A) and (L, B) for which the cohomology homomorphisms induced by inclusion, $\tilde{H}^*(K) \rightarrow \tilde{H}^*(A)$ and $\tilde{H}^*(L) \rightarrow \tilde{H}^*(B)$, both fail to be epimorphisms, yet $(K \times B) \cup (A \times L)$ is a retract of $K \times L$. The example we shall give is derived from the example described in §5. Indeed, as our next theorem shows, the problems treated in §5 are closely related to those dealt with here.

THEOREM (6.2). *If $A * (L/B)$ is not a retract of $K * (L/B)$, then $(K \times B) \cup (A \times L)$ is not retractile in $K \times L$.*

Proof. Suppose $A * (L/B)$ is not a retract of $K * (L/B)$. Then by (2.3a), $S(A \wedge (L/B))$ is not a retract of $S(K \wedge (L/B))$, so by (3.1), $A \wedge (L/B)$ is not retractile in $K \wedge (L/B)$. Therefore by (2.3d) and (3.3d), the identification map $q: K \wedge (L/B) \rightarrow (K/A) \wedge (L/B)$ induces a homomorphism $q^*: [(K/A) \wedge (L/B); X] \rightarrow [(K/A) \wedge L; X]$ that fails to be 0-monic for some space X . Now, referring to the proof of (6.1), we see that $p^*: [(K/A) \wedge (L/B); X] \rightarrow [K \times L; X]$ fails to be 0-monic for this same space X ; hence $(K \times B) \cup (A \times L)$ is not retractile in $K \times L$.

COROLLARY (6.3). If the homomorphism $i^*: \tilde{H}^n(K; \tilde{H}^k(L, B)) \rightarrow \tilde{H}^n(A; \tilde{H}^k(L, B))$ induced by the inclusion map $i: A \rightarrow K$ fails to be an epimorphism for some pair (n, k) of integers, then $(K \times B) \cup (A \times L)$ is not a retract of $K \times L$.

The proof is immediate from (5.1) and (6.2).

As to the example, let (K, A) and B be the same as in the example of §5. Thus K is a contractible complex, A is a pseudo-projective space of homology type (Z_r, m) , and B is a pseudo-projective space of homology type (Z_s, n) where m, n, r, s are integers greater than 1 and r, s are relatively prime. Let L be a finite, contractible complex obtained by adjoining cells to B . From the exact homology sequences of the pairs (K, A) and (L, B) it is easily seen that K/A and L/B are spaces of homology types $(Z_r, m+1)$ and $(Z_s, n+1)$, respectively. Hence by [5, §5] the inclusion map of $(K/A) \vee (L/B)$ into $(K/A) \times (L/B)$ is a homotopy equivalence; consequently $(K/A) \wedge (L/B)$ is a contractible complex. Then it follows from (2.3c) that the inclusion map of $(K \times B) \cup (A \times L)$ into $K \times L$ is a homotopy equivalence. Therefore $(K \times B) \cup (A \times L)$ is a retract of $K \times L$.

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